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1980 J. Phys. A: Math. Gen. 13 187

(http://iopscience.iop.org/0305-4470/13/1/020)

J. Phys. A: Math. Gen., **13** (1980) 187–189. Printed in Great Britain

On a generalisation of Birkhoff's theorem

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Received 11 January 1979

Abstract. The space–time metric is proved to be static for a broad class of physical systems within four types of symmetry.

1. Introduction

In relativistic theories of gravity it is of essential interest to know the conditions under which dynamical systems are necessarily reduced to static ones. In general relativity the spherically symmetric vacuum field is known to be static (Birkhoff's theorem, Birkhoff 1923 p 253). This result has been generalised for some special forms of the energy–momentum tensor T_μ^ν (Das 1960, Isaev 1976, Bronnikov *et al* 1976), some scalar–tensor theories (Reddy 1973, Krori and Nandy 1977)† and, moreover, for planar and pseudospherical symmetries (Ruban 1977, for electromagnetic field and $\Lambda \neq 0$ only). It seems useful to try to extend this fundamental result to broader classes of symmetries and physical systems. Evidently in such a treatment some additional limitations can arise (e.g. in the case of cylindrical symmetry even vacuum solutions are, in general, wavelike, see Einstein and Rosen 1937).

In this paper we generalise the theorem in two respects: (i) four types of symmetries are considered, and (ii) some general requirements upon T_μ^ν which provide the static character of the metric, are formulated. Finally, some physical systems satisfying these requirements, are listed.

2. Theorem

The metric is taken in the general form

$$\mathrm{d}s^2 = \mathrm{e}^{2\gamma}\,\mathrm{d}T^2 - \mathrm{e}^{2\lambda}\,\mathrm{d}R^2 - \mathrm{e}^{2\mu}\,\mathrm{d}\eta^2 - \mathrm{e}^{2\beta} f^2(\eta)\,\mathrm{d}\xi^2, \qquad (1)$$

where γ, λ, μ, β are functions of R and T. It corresponds to the following symmetries considered: cylindrical ($f \equiv 1$, μ and β arbitrary), planar ($f \equiv 1$, $\mu \equiv \beta$), spherical ($f = \sin\eta$, $\mu \equiv \beta$) and pseudospherical ($f = \sinh\eta$, $\mu \equiv \beta$).

The following components of the Einstein tensor G_μ^ν are of interest for us (here and henceforth a dot denotes $\partial/\partial T$, a prime $\partial/\partial R$):

$$G_0^0 = \mathrm{e}^{-2\gamma}(-\dot\lambda\dot\mu - \dot\lambda\dot\beta - \dot\mu\dot\beta) + \mathrm{e}^{-2\lambda}(\mu'' + \beta'' + \mu'^2 + \beta'^2 - \lambda'\mu' - \lambda'\beta' + \mu'\beta') - \epsilon\,\mathrm{e}^{-2\beta}, \qquad (2)$$

$$G_1^1 = \mathrm{e}^{-2\gamma}(-\ddot\mu - \ddot\beta - \dot\mu^2 - \dot\beta^2 + \dot\gamma\dot\mu + \dot\gamma\dot\beta - \dot\mu\dot\beta) + \mathrm{e}^{-2\lambda}(\gamma'\mu' + \gamma'\beta' + \mu'\beta') - \epsilon\,\mathrm{e}^{-2\beta}, \qquad (3)$$

† The paper by Reddy contained an error which was corrected by Krori and Nandy.

$$G_0^1 = e^{-2\lambda}[-\dot{\mu}' - \dot{\beta}' - \dot{\mu}\mu' - \dot{\beta}\beta' + \gamma'(\dot{\mu} + \dot{\beta}) + \dot{\lambda}(\mu' + \beta')], \tag{4}$$

where $\epsilon = 1$ ($\epsilon = -1$) for (pseudo) spherical symmetry and $\epsilon = 0$ for planar and cylindrical ones.

We adopt the following conditions:

(1). The Einstein equations $G_\mu^\nu = -\kappa T_\mu^\nu$ are fulfilled.

(2). There exists a functional relation between $\mu(R, T)$ and $\beta(R, T)$ (it is manifestly so, except for cylindrical symmetry).

(3). In the frame of reference such that $\dot{\mu} = \dot{\beta} = 0$ (its existence is guaranteed by (2)), the two conditions upon T_μ^ν are valid: (i) $T_0^1 = 0$ and (ii) there exists a linear combination T_1^1 + constant T_0^0 which does not contain $\gamma(R, T)$ and material quantities depending explicitly on T.

With these requirements satisfied, the metric is static (the generalised Birkhoff theorem).

The additional requirement (2) for cylindrical symmetry is necessary in order to eliminate the wave solutions.

A proof can be outlined as follows. The Einstein equation $\binom{1}{0}$ combined with (3(i)) leads to either $\dot{\lambda} = 0$, or $\beta' + \mu' = 0$. The second possibility is considered as an exotic case and discussed separately.

Taking into account (3(ii)), with $\dot{\lambda} = \dot{\mu} = \dot{\beta} = 0$, we see that a combination G_1^1 + constant G_0^0 depends only on R. Since this expression contains $\gamma(R, T)$ only in the form $\gamma'(\mu' + \beta')$ and $\mu' + \beta' \neq 0$, one concludes that $\gamma' = \gamma'(R)$, whence $\gamma = \gamma(R)$ for a proper choice of the time coordinate. This completes the proof.

One should note that the general form of metric (1) ($f(\eta)$ arbitrary) is automatically reduced to the four symmetries considered if $T_2^0 = T_2^1 = 0$.

3. Examples

The following material sources of gravity satisfy the requirements of the theorem:

(1) Static massless scalar field (S): $\psi = \psi(R)$.

(2) Cosmological massless scalar field (S_T): $\psi = \psi(T)$.

(3) Three types of Maxwell fields: radial (R), longitudinal (L), and azimuthal (A). (L- and A-fields exist only within cylindrical symmetry).

(4) Arbitrary massless gauge fields with T_μ^ν similar to that of the Maxwell field (R-type).

(5) Static massive scalar field (S^m).

(6) Massive vector (V) field A_μ with $A_\alpha = \delta_\alpha^0 A_0$, $e^{-\gamma}A_0 = A(R)$.

(7) The same fields with certain types of nonlinearities added, e.g., scalar fields with a self-interaction of the form $\Psi(\psi)$, $\psi = \psi(R)$, and some nonlinear electromagnetic R-fields (e.g. those with the Lagrangian $\Phi(F_{\alpha\beta}F^{\alpha\beta})$, in particular, the Born–Infeld field).

(8) Perfect fluid (F) with some special equations of state (e.g. p = constant ϵ), when the frame of reference $\dot{\mu} = \dot{\beta} = 0$ is comoving.

The generalised Birkhoff theorem is valid as well in cases when material sources coexist in certain combinations, e.g.:

$$(S, R, V), (S, R, F), (S, A), (S, L), (S_T, R), (S_T, L), (S_T, A).$$

Moreover, fields may exist in the form of multiplets (e.g. Higgs' scalar multiplets).

The cosmological constant Λ can be included without consequence.

Moreover, one can extend all the results to a broad class of scalar–tensor theories with the Lagrangian

$$L = A(\phi)R + B(\phi)g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - 2\Lambda(\phi) + L_m$$

with arbitrary functions A, B and Λ and matter Lagrangian L_m, under the additional requirement $\phi = \phi(R)$. This class includes, e.g., the Brans–Dicke theory and conformal scalar field in general relativity. A proof is carried out using Wagoner's (1970) transformation.

Finally, the 'exotic' assumption $\mu' + \beta' = 0$ implies that the Einstein equations are inconsistent for all the above forms of matter, except for the R-field. In the latter case β = constant and μ = constant. The Einstein equations lead to limitations upon the constants and a wave-type equation (in general, a nonlinear one) connecting $\gamma(R, T)$ and $\lambda(R, T)$.

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